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LETTER TO THE EDITOR

Critical behaviour of integrable mixed-spin chains

S R Aladim†‡ and M J Martins†

† Departamento de Física, Universidade Federal de São Carlos, São Carlos, C P 676, 13560 São Paulo, Brazil

‡ Instituto de Física e Química de São Carlos, São Carlos, 13560 São Paulo, Brazil

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Abstract. We construct a mixed spin-$\frac{1}{2}$ and spin-S integrable model and investigate its finite-size properties. For a certain conformal invariant mixed spin system the central charge can be decomposed in terms of the conformal anomaly of two single integrable models of spin $\frac{1}{2}$ and spin $(S-\frac{1}{2})$. We also compute the ground-state energy and the sound velocity in the thermodynamic limit.

Integrable magnetic spin chains provide important examples of systems which can be derived from the so-called Yang–Baxter algebra [1]. A well known model is the isotropic spin-$\frac{1}{2}$ Heisenberg [2] chain and its generalization for arbitrary spin S [3, 4]. Another interesting example is the Heisenberg model in the presence of an impurity of spin S [5, 6]. In such a model one of the local vertex weights acts on a pair of asymmetric vector spaces which is defined by the local states on the horizontal and vertical lines of a two-dimensional lattice. More recently, a general discussion concerning the construction of mixed-vertex models has been presented by de Vega and Woynarovich [7]. For instance, they have studied several properties of the thermodynamic limit of an alternating anisotropic chain of spins $\frac{1}{2}$ and 1. However, the finite-size effects in these mixed-spin models as well as their class of universality for conformally invariant systems is still to be investigated. Following the approach of [7] we construct an isotropic alternating spin-$\frac{1}{2}$ and spin-S chain. We focus our attention in the analysis of the finite-size behaviour of the ground state on a line of length L. A conformally invariant mixed system can be defined and its conformal anomaly is computed by analysing the finite-size corrections for the ground-state energy and by the thermodynamic Bethe ansatz. Several useful quantities such as the ground-state energy and the sound velocity are also computed.

The construction of the transfer matrix of the mixed spin-σ and spin-S model is based on the local vertex $R^{\sigma}_{S,j}(\lambda)$ which is a matrix in the auxiliary space V_σ and its matrix elements are operators of spin S acting on the Hilbert state space at site j. In the case of an auxiliary space of spin $\frac{1}{2}$, $R^{1/2}_{S,j}(\lambda)$ [8] is given by

$$R^{1/2}_{S,j}(\lambda) = \begin{pmatrix} \lambda S^0 + \mathrm{i}S^3 & \mathrm{i}S^- \\ \mathrm{i}S^+ & \lambda S^0 - \mathrm{i}S^3 \end{pmatrix} \qquad S^{\pm}_j = S^1_j \pm \mathrm{i}S^2_j \tag{1}$$

where S^a_j, $a = 1, 2, 3$ are spin-S operators and S^0 is a $(2S+1)\times(2S+1)$ identity matrix.

In the case of an alternating spin-$\frac{1}{2}$ and spin-S mixed model the set of commuting transfer matrix $T_{1/2,S}(\lambda)$ assumes the following form:

$$T_{1/2,S}(\lambda) = Tr_{V_{1/2}}(\tau_{1/2,S}(\lambda)) \qquad \tau_{1/2,S}(\lambda) = R^{1/2}_{1/2,L}(\lambda) R^{1/2}_{S,L-1}(\lambda) \cdots R^{1/2}_{S,1}(\lambda) \tag{2}$$

where $\tau_{1/2,S}$ is a (2×2) matrix in the auxiliary space $V_{1/2}$ denominated the monodromy matrix. Here we impose periodic boundary conditions and the length L is an even number.

The associated one-dimensional quantum Hamiltonian which commutes with $T_{1/2,S}$ is defined by $H_{1/2,S} = \mathrm{i}J\,\mathrm{d}/\mathrm{d}\lambda \log(T_{1/2,S}(\lambda))|_{\lambda=i/2}$ and has the following expression:

$$H_{1/2,S} = \tilde{J}\left[\sum_{n=\text{even}}^{L}\sum_{i,j=0}^{3}\sigma_{n-1}^{i}\{S_n^i, S_n^j\}\sigma_{n+1}^{j} + \sum_{i=1}^{3}\left(\frac{1}{4} - S(S+1)\right)\sigma_{n-1}^{i}\sigma_{n+1}^{i}\right] - \frac{\tilde{J}L(2S+3)^2}{8} \tag{3}$$

where σ^i, $i = 1,2,3$ are Pauli matrix elements, σ^0 is the identity matrix, and $\tilde{J} = 2J/(2S+1)^2$. In this paper we are interested in the antiferromagnetic ($J > 0$) regime of (3), and we assume, for the sake of simplicity, $J = 1$.

Similarly to the usual spin-$\frac{1}{2}$ Heisenberg chain, the Hamiltonian (3) can be diagonalized by the quantum inverse scattering method [9]. The eigenenergies are parametrized in terms of the complex numbers λ_j:

$$E_{1/2,S} = -\sum_{j=1}^{M}\frac{1}{\lambda_j^2 + \frac{1}{4}} \tag{4}$$

where λ_j satisfy the so-called Bethe ansatz equation

$$\left(\frac{\lambda_j - \mathrm{i}/2}{\lambda_j + \mathrm{i}/2}\right)^{L/2}\left(\frac{\lambda_j - \mathrm{i}S}{\lambda_j + \mathrm{i}S}\right)^{L/2} = -\prod_{l=1}^{M}\frac{\lambda_j - \lambda_l - \mathrm{i}}{\lambda_j - \lambda_l + \mathrm{i}} \tag{5}$$

where $M = \frac{1}{2}L(S + \frac{1}{2}) - r$, and r labels the disjoint sectors of the eigenvalues of the total spin operator $\sum_{j=\text{odd}}^{M}(\sigma_j) + \sum_{j=\text{even}}^{M}(S_j)$.

In order to find the structure of the numbers λ_j for finite-size systems we numerically solve (4) and (5) and compare them with the exact diagonalization of the Hamiltonian (3). In figure 1(*a*), (*b*) we show the picture of the ground state for $L = 6, 8, 10$ and $S = 1, \frac{3}{2}$. In table 1 we present the respective values of the complex parameter λ_j. For large enough $L(L \geqslant 8)$ the numbers λ_j cluster in two distinct sets of roots. One of them consists of real numbers (full circles) and the other of complex structures (crosses) λ_j^{α} which in the asymptotic limit $L \to \infty$ are called $2S$-strings

$$\lambda_j^{\alpha} = \xi_j + \tfrac{1}{2}\mathrm{i}(2S + 1 - 2\alpha) \qquad \alpha = 1, 2, \ldots, 2S \tag{6}$$

where ξ_j is a real number denominated centre of the $2S$-string.

Taking into account the structure discussed above, the ground-state energy per particle $e_{\infty}^{1/2,S}$ can be calculated and is given by

$$e_{\infty}^{1/2,s} = -\ln(2) - \frac{1}{2}\left[\psi\left(\frac{2S+3}{4}\right) - \psi\left(\frac{2S+1}{4}\right)\right] \tag{7}$$

where $\psi(x)$ is the Euler psi function.

Analogously, it is possible to define the transfer matrix $T_{S,1/2}(\lambda)$ of a mixed spin-S and spin-$\frac{1}{2}$ model which commutes with $T_{1/2,S}(\lambda)$. The transfer matrix $T_{S,1/2}(\lambda)$ has the following expression:

$$\begin{aligned} T_{S,1/2}(\lambda) &= Tr_{V_S}(\tau_{S,1/2}(\lambda)) \\ \tau_{S,1/2}(\lambda) &= R_{1/2,L}^{S}(\lambda)R_{S,L-1}^{S}(\lambda)\ldots R_{S,1}^{S}(\lambda)\,. \end{aligned} \tag{8}$$

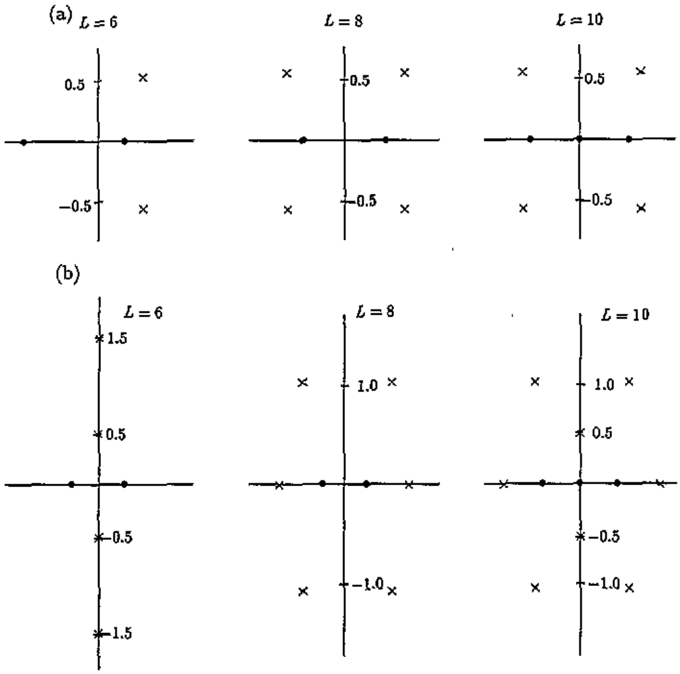

Figure 1. The ground-state configurations of the numbers λ_j for $L = 6, 8, 10$ and $S = 1, \frac{3}{2}$.

Table 1. The numerical values of the numbers λ_j of the Bethe ansatz equation (5) corresponding to the ground state for $L = 6, 8, 10$ and $S = 1, \frac{3}{2}$.

L	$S = 1$	$S = \frac{3}{2}$
6	$0.316194 \pm i\,0.508074$ $-0.494077,\ 0.228965$	$\pm i\,1.5,\ \pm i\,0.5$ ± 0.358639
8	$\pm 0.308362 \pm i\,0.503329$ ± 0.308005 $\pm 0.535111 \pm i\,0.510182$	$\pm 0.302413 \pm i\,1.060323$ $\pm 0.436072,\ \pm 0.201077$ $\pm 0.302415 \pm i\,1.060320$
10	$\pm 0.444828,\ 0$	$\pm 0.436069,\ \pm i\,0.5$ ± 0.201079

The commutativity between $T_{1/2,S}(\lambda)$ and $T_{S,1/2}(\lambda)$ derives from the Yang–Baxter relations satisfied by the local vertices $R^{g}_{S,j}$ [8]. Interestingly enough, this permits us to define a *rotational* invariant mixed vertex model by formally multiplying the two transfer matrix [7]

$$T^{\text{sym}} = T_{1/2,S}T_{S,1/2}. \tag{9}$$

Due to commutativity $[T_{1/2,S}(\lambda), T_{S,1/2}\lambda'] = 0$, the eigenenergies of the one-dimensional Hamiltonian associated to T^{sym} are parametrized by the same Bethe equations, namely (5). However, the expression for the total energy for a given set of numbers λ_j is

$$E^{\text{sym}} = -\sum_{j=1}^{M} \frac{1}{\lambda_j^2 + \frac{1}{4}} - \sum_{j=1}^{M} \frac{2S}{\lambda_j^2 + S^2}. \tag{10}$$

The model defined by (5) and (10) possesses all features of being conformally invariant, i.e. short-range interaction, translational and *rotational* symmetry and gapless low-lying excitations in the spectrum. Indeed, for low (total) momenta p†, the dispersion relation is linear in p:

$$\epsilon(p) \sim v_s p \tag{11}$$

where $v_s = 2\pi$ is the sound velocity.

The class of universality of this model can be determined by exploiting a set of important relations [10] between the eigenspectrum of finite-lattice systems. In particular the conformal anomaly is related to the ground-state energy $E_0^{\mathrm{sym}}(L)$ by [11, 12]

$$E_0^{\mathrm{sym}}(L)/L = e_\infty^{\mathrm{sym}} - \frac{\pi c v_s}{6L^2} \tag{12}$$

where c is the central charge and e_∞^{sym} is the ground state per particle

$$e_\infty^{\mathrm{sym}} = e_\infty^{1/2,S} - \frac{1}{2}\left[\psi\left(\frac{2S+3}{4}\right) - \psi\left(\frac{2S+1}{4}\right) + \psi\left(\frac{2S+1}{2}\right) - \psi\left(\frac{1}{2}\right)\right]. \tag{13}$$

In table 2 we present our estimates for the central charge c of (12) for $S = 1, \frac{3}{2}$. Our numerical result predicts a conformal anomaly $c = 2.01(2)$ ($S = 1$) and $c = 2.500(6)$ ($S = \frac{3}{2}$). We notice that for $S = 1$ the extrapolation is less precise compared with $S = \frac{3}{2}$. This is due to the fact that the bulk of the complex part of the two-string zeros are next to $\pm i/2$. As a consequence, we can apply an analytical technique developed by de Vega and Woynarovich [13] and we find the exact value $c = 2$ for $S = 1$.

Table 2. The estimates of the conformal anomaly of (12) for $S = 1$ and $S = \frac{3}{2}$.

L	$S = 1$	$S = \frac{3}{2}$
8	2.210 923	2.839 364
16	2.061 249	2.602 543
24	2.041 930	2.556 301
32	2.020 828	2.538 530
40	2.017 593	2.529 503
48	2.014 692	2.524 142
56	2.012 134	2.520 624
Extrapolated	2.01(1)	2.500(6)

Another efficient method to compute the conformal anomaly is by analysing the low-temperature behaviour of the associated free energy. For a critical system at low temperature, the free energy per particle has the following asymptotic behaviour [11, 12]:

$$F(T)/L = e_\infty - \frac{\pi c T^2}{6 v_s}. \tag{14}$$

In the case of integrable one-dimensional spin chains the thermodynamic properties can be studied by using the thermodynamic Bethe ansatz method. In this approach, the free

† We remark here that the correct momentum operator is half that used in [7] (see (2.35)). This implies that the sound velocity is double ($v_s = 2\pi$) of that found previously in [7] ($v_s = \pi$).

energy is given in terms of variables denominated pseudoenergies, and its minimization yields a set of integral equations for these parameters. In order to obtain such equations for the mixed-spin model defined by (5), and (10) we follow [3, 4] and here we give only the final results. The free energy at temperature T is given by

$$F^{\mathrm{sym}}(T)/L = e_{\infty}^{\mathrm{sym}} - \frac{T}{4\pi}\int_{-\infty}^{+\infty} p(\lambda)\left(\ln(1+\mathrm{e}^{\epsilon_1(\lambda)/T}) + \ln(1+\mathrm{e}^{\epsilon_{2S}(\lambda)/T}\right) \tag{15}$$

where the pseudoenergies $\epsilon_a(\lambda), a = 1, 2, 3, \ldots$ satisfy the following TBA equations:

$$\epsilon_n(\lambda)/T = p(\lambda) * \left(\ln(1+\mathrm{e}^{\epsilon_{n+1}/T}) + \ln(1+\mathrm{e}^{\epsilon_{n-1}/T})\right) + \frac{p(\lambda)}{T}(\delta_{n,1} + \delta_{n,2S}) \tag{16}$$

where $\epsilon_0(\lambda) = 0$, $f \times g(x)$ denotes the convolution $(1/2\pi)\int_{-\infty}^{+\infty} f(x)g(x-y)\,\mathrm{d}y$, and $p(\lambda) = \pi/\cosh(\pi\lambda)$.

The advantage of (15) and (16) is that they allow us to estimate exactly the low-temperature behaviour of the free energy. Using the standard procedure (see e.g. [4, 15]) to compute the leading behaviour of $F(T)$, and some dilogarithm identities [16] we find the following result:

$$F(T)/L = e_{\infty} - \frac{4S-1}{6(2S+1)}T^2. \tag{17}$$

Comparing (17) and (14), the value of the central charge is given by

$$c = \frac{2(4S-1)}{2S+1}. \tag{18}$$

For $S = 1, \frac{3}{2}$ we obtain $c = 2, 2.5$ which are consistent with our numerical findings of table 2. Interestingly enough, (18) can be decomposed in terms of the central charges of the integrable Heisenberg chains of spins $\frac{1}{2}$ ($c = 1$) and $S - \frac{1}{2}$ ($c = 3(2S-1)/2S+1$). In this sense the effect of the interaction between spins $\frac{1}{2}$ and S is the 'reduction' of the critical behaviour of a spin S to $S - \frac{1}{2}$†. Taking into account this last observation it is easy to conjecture the conformal anomaly of a mixed spin S and S'. In general, we can assume $S' > S$ and the expected critical behaviour is given by the following conformal anomaly:

$$c = \frac{3S}{S+1} + \frac{3(S'-S)}{S'-S+1}. \tag{19}$$

We believe that result (19) is the first step toward the understanding of the composition of the operator content of these mixed spin models. However, it is still to be investigated how the primary fields are composed in these systems. We hope to report on this problem in a future publication.

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† We remark that $J = S - \frac{1}{2}$ is the smallest possible value in the addition of two 'angular momenta', S and $\frac{1}{2}$.

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